Noncommutative oscillator, symmetry and Landau problem

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Isotropic oscillator on a plane is discussed where both the coordinate and momentum space are considered to be noncommutative. We also discuss the symmetry properties of the oscillator for three separate cases when both the noncommutative parameters Θ and $\overline{\Theta}$ satisfy specific relations. We compare the Landau problem with the isotropic oscillator on noncommutative space and obtain a relation between the two noncommutative parameters with the magnetic field of the Landau problem.

I. INTRODUCTION

In recent years noncommutative geometry [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] has received much attention due to the fact that spacetime may be noncommutative at very small length scale. For detail study on noncommutative space see the list of Refs. of Refs. [11, 12]. Although the noncommutative scale is expected to be pretty small, perhaps below the Planck scale [13], people are looking for phenomenological consequences of the noncommutative geometry in low energy quantum mechanical regime. One such example is the fractional angular momentum quantum number and the existence of the zero point expectation value of the angular momentum in noncommutative space, which has been discussed in a recent paper [14].

Usually only the coordinate space is considered to be noncommutative when the noncommutative physics is studied. But in this letter we consider both the coordinate and momentum space to be noncommutative. However momentum space non-commutativity is not a new concept. As we know in quantum mechanics the generalized momentum components are noncommutative, which can be taken as a motivation for us to study the quantum mechanical model in noncommutative momentum space. Also recently it has been shown in Ref. [14] that in order to keep the Bose Einstein statistics intact at the noncommutative level the momentum space needs to be noncommutative besides the noncommutative coordinate space.

This letter is organized as follows: In section II we consider the isotropic $2D$ oscillator in a space where both the coordinate and momentum are noncommutative. The spectrum of the system is found and its symmetry is discussed for a specific noncommutative parameters range. In section III the model of section II is compared with the Landau problem and a relation between the noncom-

mutative parameters Θ and $\overline{\Theta}$ with the magnetic field is established. Finally section IV is devoted to a discussion.

II. ISOTROPIC OSCILLATOR IN NONCOMMUTATIVE SPACE

We consider both the position and the momentum space to be noncommutative. The noncommutative algebra in the phase space can be written as (in $\hbar = 1$ unit)

$$
\begin{aligned}
[\overline{x_i}, \overline{x_j}] &= 2i\epsilon_{ij}\Theta, \quad [\overline{p_i}, \overline{p_j}] = 2i\epsilon_{ij}\overline{\Theta}, \\
[\overline{x_i}, \overline{p_j}] &= i\delta_{ij}\left[1 + \Theta\overline{\Theta}\right],
\end{aligned} \tag{1}
$$

where Θ is the noncommutative parameter for the coordinate space, $\overline{\Theta}$ is the noncommutative parameter for the momentum space and $\epsilon_{11} = \epsilon_{22} = 0, \epsilon_{12} = -\epsilon_{21} = 1$. The fact that the noncommutative parameter $\overline{\Theta}$ is not a magnetic field will be discussed later in this paper. Basically one can see that our algebra (1) is unaffected by the electric charge e of the quantum particle. But in case of magnetic field the algebra should be affected by the limit $e = 0$. The 2D isotropic oscillator Hamiltonian

$$
H = \frac{1}{2m}\boldsymbol{p}^2 + \frac{1}{2}m\omega^2\boldsymbol{r}^2, \tag{2}
$$

can now be defined as the same Hamiltonian but now the coordinate and momentum are replaced by the corresponding noncommutative counter parts. The noncommutative counterpart of the above Hamiltonian is [8, 12]

$$
\overline{H} = \frac{1}{2m}\overline{\boldsymbol{p}}^2 + \frac{1}{2}m\omega^2\overline{\boldsymbol{r}}^2. \tag{3}
$$

It is possible to go back to the commutative space by replacing the noncommutative operators in terms of the commutative operators. We therefore need to get a representation of the algebra (1) in terms of the commutative coordinates. One possible representation is given by

$$
\begin{aligned}
\overline{x_1} &= x_1 - \Theta p_2, \quad \overline{x_2} = x_2 + \Theta p_1, \\
\overline{p_1} &= p_1 + \overline{\Theta}x_2, \quad \overline{p_2} = p_2 - \overline{\Theta}x_1,
\end{aligned} \tag{4}
$$

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The above representation is consistent with the algebra (1) and can be used to go back to commutative space, where the problem can be handled easily. Our Hamiltonian (2) thus becomes

$$H_{\Theta,\overline{\Theta}} = \frac{1}{2M_\Theta}\boldsymbol{p}^2 + \frac{1}{2}M_\Theta\Omega_{\Theta,\overline{\Theta}}^2\boldsymbol{r}^2 - S_{\Theta,\overline{\Theta}}L_z\,, \qquad (5)$$

where $1/M_\Theta = 1/m + m\omega^2\Theta^2$, $\Omega_{\Theta,\overline{\Theta}} = \sqrt{(1/m + m\omega^2\Theta^2)\left(m\omega^2 + \overline{\Theta}^2/m\right)}$ and $S_{\Theta,\overline{\Theta}} = m\omega^2\Theta + \overline{\Theta}/m$.

This Hamiltonian can be solved exactly. It is possible to write the Hamiltonian (5) as a sum of two separate 1D harmonic oscillators with frequencies $\Omega_{\Theta,\overline{\Theta}}^+, \Omega_{\Theta,\overline{\Theta}}^-$ [15, 16, 17, 18],

$$H_{\Theta,\overline{\Theta}} = \Omega_{\Theta,\overline{\Theta}}^+\left(a_{\Theta,\overline{\Theta}}^\dagger a_{\Theta,\overline{\Theta}} + 1/2\right) + $$
$$\Omega_{\Theta,\overline{\Theta}}^-\left(b_{\Theta,\overline{\Theta}}^\dagger b_{\Theta,\overline{\Theta}} + 1/2\right) \qquad (6)$$

where the annihilation operators $a_{\Theta,\overline{\Theta}}, b_{\Theta,\overline{\Theta}}$,

$$a_{\Theta,\overline{\Theta}} = \frac{1}{2\sqrt{M_\Theta\Omega_{\Theta,\overline{\Theta}}^+}}\left[(p_1 + ip_2) - iM_\Theta\Omega_{\Theta,\overline{\Theta}}^+(x_1 + ix_2)\right] (7)$$

$$b_{\Theta,\overline{\Theta}} = \frac{1}{2\sqrt{M_\Theta\Omega_{\Theta,\overline{\Theta}}^-}}\left[(p_1 - ip_2) - iM_\Theta\Omega_{\Theta,\overline{\Theta}}^-(x_1 - ix_2)\right](8)$$

and the corresponding creation operators $a_{\Theta,\overline{\Theta}}^\dagger, b_{\Theta,\overline{\Theta}}^\dagger$ satisfy the usual commutation relations

$$\left[a_{\Theta,\overline{\Theta}}, a_{\Theta,\overline{\Theta}}^\dagger\right] = \left[b_{\Theta,\overline{\Theta}}, b_{\Theta,\overline{\Theta}}^\dagger\right] = 1\,, \qquad (9)$$

with all other commutators being zero. The explicit form of the frequencies are given by

$$\Omega_{\Theta,\overline{\Theta}}^+ = \sqrt{\Omega_{\Theta,\overline{\Theta}}^2 - S_{\Theta,\overline{\Theta}}^2} + S_{\Theta,\overline{\Theta}}\,, \qquad (10)$$

$$\Omega_{\Theta,\overline{\Theta}}^- = \sqrt{\Omega_{\Theta,\overline{\Theta}}^2 - S_{\Theta,\overline{\Theta}}^2} - S_{\Theta,\overline{\Theta}}\,, \qquad (11)$$

where $\Omega_{\Theta,\overline{\Theta}}^+ \neq \Omega_{\Theta,\overline{\Theta}}^-$. The number operators $\mathcal{N}_{\Theta,\overline{\Theta}}^+ = a_{\Theta,\overline{\Theta}}^\dagger a_{\Theta,\overline{\Theta}}$ and $\mathcal{N}_{\Theta,\overline{\Theta}}^- = b_{\Theta,\overline{\Theta}}^\dagger b_{\Theta,\overline{\Theta}}$ satisfy the eigenvalue equation $\mathcal{N}_{\Theta,\overline{\Theta}}^\pm|n^+, n^-\rangle = n^\pm|n^+, n^-\rangle$ with $n^\pm = 0, 1, 2, 3, \dots$. Now the exact eigenvalue for the Hamiltonian (6) is known in literature and is of the form

$$E_{\Theta,\overline{\Theta}} = \Omega_{\Theta,\overline{\Theta}}^+(n^+ + 1/2) + \Omega_{\Theta,\overline{\Theta}}^-(n^- + 1/2)\,, \qquad (12)$$

which is non-degenerate due to the presence of non-commutativity. But it is also possible to get back the $su(2)$ symmetry for a specific case when the ratio of the two frequencies is rational, $\Omega_{\Theta,\overline{\Theta}}^+/\Omega_{\Theta,\overline{\Theta}}^- = a^-/a^+$, a_-, a_+ are relatively prime numbers. But before going into the complicated case we first mention the simplest case, $\overline{\Theta} = -m^2\omega^2\Theta$, when also the $su(2)$ symmetry is
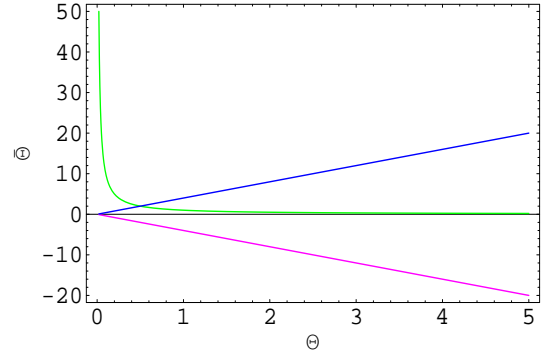

FIG. 1: (color online) A plot of $\overline{\Theta}$ as a function of Θ for $m = 1$, $\omega = 2$. The green curve corresponds to $\Theta\overline{\Theta} = 1$ (it corresponds to a spectrum given in Eq. (15), which is infinitely degenerate), the bule curve corresponds to $\overline{\Theta} = m^2\omega^2\Theta$ (it restores BE statistics) and the pink curve corresponds to $\overline{\Theta} = -m^2\omega^2\Theta$ (it corresponds to a spectrum of Eq. (13), which has $su(2)$ symmetry)

recovered. Since for $\overline{\Theta} = -m^2\omega^2\Theta$ the third term in the Hamiltonian (5) is zero the Hamiltonian becomes isotropic. So the the eigenvalue becomes

$$E_{\Theta,\overline{\Theta}=-m^2\omega^2\Theta} = \left(n^+ + n^- + 1\right)\Omega_\Theta\,, \qquad (13)$$

where the frequency is $\Omega_\Theta = \omega\left(1 + m^2\omega^2\Theta^2\right) = \omega\left(1 - \Theta\overline{\Theta}\right)$. Another important case is when

$$\Omega_{\Theta,\overline{\Theta}} = S_{\Theta,\overline{\Theta}} \Rightarrow \Theta\overline{\Theta} = 1\,. \qquad (14)$$

Due to the constraint (14) the Hamiltonian is infinitely degenerate and the eigenvalues are given by

$$E_{\Theta\overline{\Theta}=1} = (2n + 1)\widehat{\Omega_\Theta}\,, \qquad (15)$$

where $\widehat{\Omega_\Theta} = m\omega^2\Theta + \frac{1}{\Theta m}$ and $n = 0, 1, 2, \dots$ Note that the above spectrum is obtained in Ref. [16], but there the origin of momentum noncommutativity is due to magnetic field and thus coupled to the electric charge of the particle. Now let us consider the non degenerate spectrum (12) which is known [19] to have $su(2)$ symmetry in a certain case. Let us assume that for $n^\pm = a^\pm p^\pm + b^\pm$, the ratio of the two frequencies be such that, $\Omega_{\Theta,\overline{\Theta}}^+ a^+ = \Omega_{\Theta,\overline{\Theta}}^- a^- = \widehat{\Omega_{\Theta,\overline{\Theta}}}$ for certain constants $a^\pm$, $b^\pm$ and $p^\pm = 0, 1, 2, \dots$ Then the spectrum (12) becomes

$$E_{\Theta,\overline{\Theta}} = \widehat{\Omega_{\Theta,\overline{\Theta}}}(p^+ + p^-) + \Omega_{\Theta,\overline{\Theta}}^+ b^+ + \Omega_{\Theta,\overline{\Theta}}^- b^-\,, \quad (16)$$

which has $su(2)$ symmetry. Note that the $\overline{\Theta} \to 0$ limit of (16) should reduce to the results already obtained in Ref. [15].

III. LANDAU PROBLEM AND NONCOMMUTATIVE SPACE

We now consider the Landau problem on a plane. For different discussions on Landau problems see Refs. [6, 20]

The Hamiltonian in symmetric gauge $\boldsymbol{A} = B/2(-x_2, x_1)$ is given by

$$H_B = \frac{1}{2\widetilde{m}}\boldsymbol{p}^2 + \frac{B^2}{8\widetilde{m}}\boldsymbol{r}^2 - \frac{B}{2\widetilde{m}}L_z \,, \qquad (17)$$

where B is the constant magnetic field in the z direction. The eigenvalues known as Landau levels are known to be infinitely degenerate

$$E_B = (2n+1)\Omega_B \,, \qquad (18)$$

where $\Omega_B = \frac{B}{2\widetilde{m}}$. Note the similarity between the Landau levels (18) and (15). The two spectrum will overlap with each other if

$$\widehat{\Omega_\Theta} = \Omega_B \,, \text{ for } \Theta\overline{\Theta} = 1 \,. \qquad (19)$$

Due to this similarity between the two spectrum one can exploit it and get a relation between the noncommutative parameters and the magnetic field in the Landau problem. On comparing the mass term of the two Hamiltonians, namely (5) and (17), we obtain

$$M_\Theta = \widetilde{m} \,. \qquad (20)$$

Considering (19) and (20) together we get

$$\overline{\Theta} = 1/\Theta = B/2 \,. \qquad (21)$$

We used the unit where $\hbar = c = e = 1$. For the quantum hall experiment the typical strength of magnetic field is $B \sim 12T - 15T$. Then the noncommutative parameters are typically

$$\Theta = 1/\overline{\Theta} \sim 0.22 \times 10^{-11} cm - 0.176 \times 10^{-11} cm \,. \qquad (22)$$

Note that by comparing term by term of the Hamiltonian of the Landau problem and the Hamiltonian of an oscillator in noncommutative coordinate space in Ref. [21] an inverse relation between the noncommutative parameter and magnetic field has been established, which is comparable with the second and third term of Eq. (21) and numerical value of the noncommutative parameter due to coordinate non-commutativity is also obtained. One should keep in mind that the above relation (21) is just an analogy but in general there is now relation of the two noncommutative parameters Θ and $\overline{\Theta}$ with the magnetic field. Although from the representation of noncommutative momentum given in Eq. (4) one may think that it is analog to the generalized momentum $\boldsymbol{P} = \boldsymbol{p} - e\boldsymbol{A}$ in constant magnetic filed background written in symmetric gauge for the vector potential $\boldsymbol{A}$ but this is clearly not so. The reason they are different will be evident if one consider neutral particle for which the noncommutative momentum still satisfy the commutative relation (1) but in magnetic field background now the generalized momentum $\boldsymbol{P} = \boldsymbol{p}$ (because $e = 0$), which means they commute. In fact the momentum noncmmutativity is used for the neutral particle model of neutron in a quantum gravitational well [22] and bounds for the noncommutative parameters have been obtained using the experimental result [23] of neutron gravitational states.
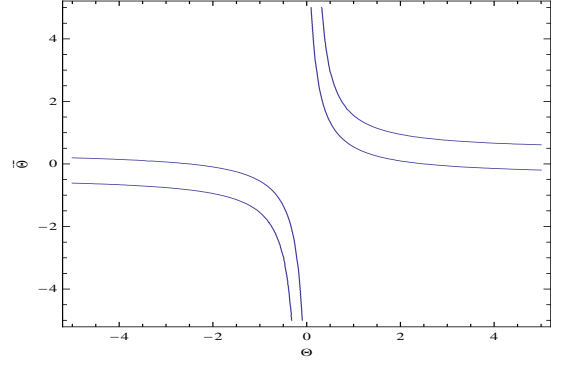

FIG. 2: (color online) A contour plot of $\Omega^+_{\Theta,\overline{\Theta}}/\Omega^-_{\Theta,\overline{\Theta}} = a^+/a_-$ for $a_+ = 2$, $a_- = 3$, $m = 1$, $\omega = 2$ is drawn. Along the curve the $su(2)$ symmetry is restored. The spectrum along the curve is given by Eq. (16) with $a_+ = 2$, $a_- = 3$, $m = 1$, $\omega = 2$

IV. CONCLUSION AND DISCUSSIONS

Isotropic oscillator on a plane is considered in this article, where we take both the coordinate and momentum space to be noncommutative. We discussed its general solution which is non degenerate. But it is possible to get back the degenerate spectrum when the noncommutative parameters Θ and $\overline{\Theta}$ satisfy certain constrants. For example for $\overline{\Theta} = -m^2\omega^2\Theta$ the system is $su(2)$ symmetric, the spectrum being given by Eq. (13). Note that for the condition $\overline{\Theta} = -m^2\omega^2\Theta$ Bose-Einstein statistics does not remain intact at the noncommutative level [14], which can be understood as follows. On noncommutative space defined by the algebra (1) one can construct the deformed creation and annihilation operators $\overline{a_i} = \sqrt{\frac{1}{2m\omega}}\left(m\omega\overline{x_i} + i\overline{p_i}\right)$ and $\overline{a_i}^\dagger = \sqrt{\frac{1}{2m\omega}}\left(m\omega\overline{x_i} - i\overline{p_i}\right)$ respectively replacing the corresponding commutative coordinate and momentum by its corresponding noncommutative counterpart. For the two creation operators $\left[\overline{a_1}^\dagger, \overline{a_2}^\dagger\right] \neq 0$, for $\overline{\Theta} = -m^2\omega^2\Theta$, which indicates that BE statistics is violated. But it is also possible to keep Bose-Einstein statistics intact i.e., $\left[\overline{a_1}^\dagger, \overline{a_2}^\dagger\right] = 0$, which will lead to $\overline{\Theta} = m^2\omega^2\Theta$. It is known from Ref. [14] that the angular momentum $\overline{L} = \overline{x_1}\,\overline{p_2} - \overline{x_2}\,\overline{p_1}$ does possess zero point eigenvalue unlike commutative case where $\langle\text{g.s}|L = x_1 p_2 - x_2 p_1|\text{g.s}\rangle = 0$. In our notation $\overline{L} = \left(1 + \Theta\overline{\Theta}\right)L - m^2\omega^2\Theta\left(x_1^2 + x_2^2\right) - \Theta\left(p_1^2 + p_2^2\right)$, for $\overline{\Theta} = m^2\omega^2\Theta$. The zero point eigenvalue $\langle\text{g.s}|\overline{L}|\text{g.s}\rangle = 2m\omega\Theta$. The most general algebra satisfied by $\overline{a_i}, \overline{a_i}^\dagger$ is $\left[\overline{a_i}, \overline{a_i}^\dagger\right] = \frac{1}{2m\omega}\left[2i(m^2\omega^2\Theta + \overline{\Theta})\epsilon_{ij} + m\omega[1 + \Theta\overline{\Theta}](\delta_{ij} + \delta_{ji})\right]$, $\left[\overline{a_i}, \overline{a_i}\right] = \left[\overline{a_i}^\dagger, \overline{a_i}^\dagger\right] = \frac{1}{2m\omega}\left[2i(m^2\omega^2\Theta - \overline{\Theta})\epsilon_{ij}\right]$. It is to be noted from (1) that the Planck constant is modified like $\overline{\hbar} = \left(1 + \Theta\overline{\Theta}\right)\hbar$ due to simultaneous consideration the coordinate and momentum space noncommutativity as pointed out in Ref. [22] and in other papers also. Finally, it may be noted that instead of starting with an isotropic oscillator on the noncommutative plane, one

could have started with an anisotropic oscillator on the noncommuting plane [24]. In that case only the mass $M_{\Theta,\overline{\Theta}}$ would change and the relation concerning Θ, $\overline{\Theta}$ and B will remain unchanged.

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